An Embodied approach to the Calculus

How learners really understand

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How an embodied approach to the calculus can provide fundamental cognitive foundations for the proceptual world of symbolism and the formal world of proof.

Formal analysis begins with the limit concept. This is entirely appropriate for a top-down formal logical structure. However …

1. Starting with the limit concept does not work cognitively
2. Learning to press buttons in sequence on the TI-92 alone does not help students understand the calculus.
3. Beginning with an embodied approach can lay the foundations for both symbolic proceptual ideas and formal ideas of definition and proof.
4. Graphic software (including the TI-92) can be used to visualize embodied concepts.
5. The programming facility on the TI-92 can be used to gain insight into the processes.

[1-4 will be covered today, 5 will be considered tomorrow.]
Students’ views of the calculus
Is the limit concept the right place to start?

What happens as $B \to A$?

‘The chord line doesn’t get close to the tangent because the line is infinite; even if the lines look close, far away at infinity they are still a long way apart.’

‘The tangent can’t be a limit because you can’t get past a limit and part of the chord is already past it.’

Some students see the chord as a finite segment tending to zero length, so ‘the chord tends to the tangent’ because the vanishing chord gets closer to the tangent.

Some students see a static picture with no movement, others (quite sensibly) see $B$ move along the chord to $A$.

Other problems with limit as a process not a concept.

The notion of “$x$ getting close to $a$” gives an image of a quantity becoming arbitrarily small, but not zero. This leads to infinitesimal ideas.

The formal definition of limit has too many quantifiers and proves to be very difficult.

The limit is not a good place to start.

‘It’s typical of teachers to show us a lot of difficult methods before getting on to the easy way to do it.’
Difficulties with the meaning of calculus symbolism

What does \( \frac{dy}{dx} \) mean? Is it a ratio of dy divided by dx?

No! say most teachers.

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

In the ‘function of a function rule’:

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}
\]

\( dx \) has no separate meaning and cannot be cancelled.

In

\[
\int f(x) \, dx
\]

\( dx \) means ‘with respect to \( x \).’

In a differential equation such as

\[
\frac{dy}{dx} = -\frac{x}{y}
\]

the indivisible symbol \( \frac{dy}{dx} \) can be separated but \( dy, \ dx \) are not quantities, they are ‘with respect to…’ in integrals

\[
\int y \, dy = \int -x \, dx.
\]
A cognitive approach to the calculus

Is the limit concept intuitive?

On the graph $y=x^2$, the point $A$ is $(1,1)$, the point $B$ is $(k,k^2)$ and $T$ is a point on the tangent to the graph at $A$.

(i) Write down the gradient of the straight line through $A$, $B$.

(ii) Write down the gradient of $AT$.

Explain how you might find the gradient of $AT$ from first principles.

No-one who has not studied the calculus offered a ‘limiting’ argument for the gradient of the tangent.

A few could ‘see’ the gradient of the tangent was 2 and also express the gradient of the chord for $k \neq 1$ as

$$\frac{k^2-1}{k-1} = k+1$$

When shown that as $k$ tends to 1, so $k+1$ tends to 2, the idea appears to them as a revelation.
The Leibniz notation

Jam recta aliqua pro arbitrio assumta vocetur \( dx \), & recta quae sit ad \( dx \) ut \( y \) est ad \( XB \) vocetur \( dy \).

Now some straight line selected arbitrarily is called \( dx \), and the line which is to \( dx \) as \( y \) is to \( XB \) is called \( dy \).
Theoretical background

Theories

Reflection

Perception Action

Environment

Locally straight calculus
Locally straight calculus
AN EMBODIED APPROACH TO THE CALCULUS

What it is not:
It is *not* an approach that begins with formal ideas of limits, but with embodied ideas of graphical representations of functions.
It is *not* an approach based *only* on real-world applications. Each real world application involving say length, area, velocity, acceleration, density, weight etc has specific sensory perceptions that are *in addition* to the ideas of the calculus and therefore may cloud the issue. The important focus is on the embodied properties of the graph of a function.

What it is:
It is a study of functions involving variables that are *numbers*. (The slope is a variable *number*, the area is a *number* that can be graphed as numerical quantities.)
It is at its best when the embodied is linked to proceptual calculations (numeric and algebraic).
Where appropriate, it can be used to motivate ideas that later can be turned into axiomatic proofs.
COMPUTER ENVIRONMENTS FOR COGNITIVE DEVELOPMENT

• a generic organiser is an environment (or microworld) which enables the learner to manipulate examples and (if possible) non-examples of a specific mathematical concept or a related system of concepts. (Tall, 1989).

• a cognitive root (Tall, 1989) is a cognitive unit which is (potentially) meaningful to the student at the time, yet contain the seeds of cognitive expansion to formal definitions and later theoretical development. (Usually embodied!)

\[ f(x) = \sin x \]

Local straightness is a cognitive root for differentiation. The program Magnify is a generic organizer for it.

The original GRAPHIC CALCULUS is a free download from www.graphiccalculus.co.uk.
The blancmange: a graph which is \textit{nowhere} locally straight

\[ f(x) = \text{bl}(x) \]

A graph which nowhere looks straight

It is the sum of saw-teeth
\[ s(x) = \min(d(x), 1 - d(x)), \text{ where } d(x) = x - \text{INT}x, \]

\[ s_n(x) = s(2^{n-1}x) / 2^{n-1}. \]

\[ \text{bl}(x) = s_1(x) + s_2(x) + s_3(x) + \ldots \]

Note: In the lecture, I shall give an embodied proof that this graph is nowhere locally straight, using my hands and your imagination …
A ‘smooth-looking curve’ that magnifies ‘rough’.

\[ N(x) = \frac{bl(1000x)}{1000} \]

\( \sin x \) is differentiable everywhere

\( \sin x + n(x) \) is differentiable nowhere!
EMBODIED LOCAL STRAIGHTNESS

Using an enactive interface.

Differentiation

f) zoom in to sense the lessening curvature and establish local straightness by sensing it ‘happen’.
b) imagine looking along a locally straight graph to see its changing slope.
c) explore ‘corners’ (with different left and right slopes) and more general ‘wrinkled’ curves to sense that not all graphs are locally straight.
d) use software to draw the slope function to establish visual relationship between a locally straight function and its slope.

Eg slope of \( x^2 \) is \( 2x \), of \( x^2 \) is \( 3x^2 \), and relate to symbolic calculation.
e) Slope functions of of \( \sin x \), \( \cos x \), and ‘explain’ the minus sine in the derivative of \( \cos x \) is \( -\sin x \)…
f) Eg: Explore \( 2^x \), \( 3^x \) and vary the parameter \( k \) in \( k^x \) to find a value of \( k \) such that the slope of \( k^x \) is again \( k^x \).

A graph with a similar shaped gradient

What about \( 3^x \)? Investigate e? A natural number.
Embodied Local Straightness & Mathematical Local Linearity

‘Local straightness’ is a primitive human perception of the visual aspects of a graph. It has global implications as the individual looks along the graph and sees the changes in gradient, so that the gradient of the whole graph is seen as a global entity.

Local linearity is a *symbolic linear approximation* to the slope at a single point on the graph, having a linear function approximating the graph at that point. It is a mathematical formulation of gradient, taken first as a limit at a point $x$, and only then varying $x$ to get the formal derivative. Local straightness remains at an embodied level and links readily to the global view.

The gradient of $\cos x$ (drawn with Blokland et al (2000) available from [www.vusoft.nl](http://www.vusoft.nl).

- an ‘embodied approach’ …
- can be linked directly to numeric and graphic derivatives, as required.
- fits exactly with the notion of local straightness.
- uses enactive software to build up the concept in an embodied form.
LOCAL LINEARITY AND THE SOLUTION OF DIFFERENTIAL EQUATIONS

A generic organiser to build a solution of a first order differential equation by hand, (Blokland et al, (2000)).
Insight into genuine theoretical problems with differential equations

Solve:
\[
\frac{dy}{dx} = \frac{1}{x^2}
\]

The analytic method is to try to think of a function \( y = f(x) \) so that \( f'(x) = \frac{1}{x^2} \). This is usually given as
\[
y = -\frac{1}{x} + c.
\]

But this is not the most general solution.
An embodied solution of a first order differential equation

A ‘direction diagram’ of line segments of gradient $1/x^2$
A typical solution is traced below. In following the directions of the short line segments, *the solution curve does not cross over the y-axis.*

![Diagram](image)

A solution of the differential equation

A ‘connected solution’ describes the continuous motion of a point \((x, y)\). A solution curve is in one piece. To the left of the \(y\)-axis, it will be in the form

\[
I(x) = -1/x + c \quad \text{for } x < 0.
\]

A solution as a *function* could be

\[
I(x) = \begin{cases} 
-1/x + c & \text{if } x < 0 \\
-1/x + k & \text{if } x > 0
\end{cases}
\]

where the constants \(c\) and \(k\) may be different.
A second fundamental flaw in symbolic solutions is that not all differential equations have solutions which are given as combinations of standard functions. E.g.:

$$\frac{dy}{dx} = x^2 + y^2$$

A numerical solution of a differential equation

In the real world the theory of calculus is used to solve differential equations by numerical methods in weather-forecasting, aerodynamic design and many other areas where solutions using formulae are totally inappropriate.
Simultaneous first order differential equations

Let the point \((x,y)\) vary with time \(t\):
\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -x
\]

Imagine the point \((t,x,y)\) in three dimensions moving along a curve which has tangent direction \((dt, dx, dy)\) determined by the equations:
\[
\frac{dx}{dt} = y \ dt, \ \frac{dy}{dt} = -x \ dt.
\]
(10)
The tangent vector is
\[
(dt, dx, dy) = (dt, y \ dt, -x \ dt)
\]
which is in the direction \((1, y, -x)\).

![Figure 6: A numerical solution of simultaneous differential equations](image)

Similar interesting phenomena occur when differential equations for \(dy/dx\) are given in terms of \(dy/dt\) and \(dx/dt\).
Higher order differential equations

$$\frac{d^2 x}{dt^2} = -x$$

Unlike a first order differential equation, this does not seem to have a direction for each point \((x,y)\):

Many solutions of a second order differential equation through one point (in this case the origin)
Introduce another variable for the first derivative, \( v = \frac{dx}{dt} \), to give two simultaneous equations:

\[
\frac{dx}{dt} = v \\
\frac{dv}{dt} = -x.
\]

A solution follows the tangential direction \((dt, dx, dv)\) in three-space given by

\[
(dt, dx, dv) = (dt, vdx, -xdv).
\]

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= -x
\end{align*}
\]

A solution to \( \frac{d^2 x}{dt} = \Box x \) using the substitution \( v = \frac{dx}{dt} \) in \((t, x, v)\) space

The area as an embodied concept
Positive and negative area calculations

Area calculation with tiny steps
CONTINUITY

The blancmange graph and a rectangle to be stretched to fill the screen:

Locally straight calculus
**Embodied definition:** A real function is continuous if it can be pulled flat.

Draw the graph with pixels height 2e, imagine \((a, f(a))\) in the middle of a pixel. Find an interval \(a-d\) to \(a+d\) in which the graph lies inside the pixel height \(f(a)\pm\)

Example: \(f(x) = \sin x\) pulled flat from .999 to 1.001:

Area under \(\sin x\) from 1 to 1.001 stretched horizontally

This is the **Fundamental Theorem of Calculus** embodied. (Think about it!)
Formalizing the embodiment of the fundamental theorem.

Let \( A(x) \) be the area under a continuous graph over a closed interval \([a,b]\) from \(a\) to a variable point \(x\).

Here, ‘continuous’ means ‘may be stretched horizontally to “look flat”’. This means that, given an \( \varepsilon > 0 \), and a drawing in which the value \((x_0, f(x_0))\) lies in the centre of a practical line of thickness \( f(x_0) \pm \varepsilon \), then a value \( \delta > 0 \) can be found so that the graph over the interval from \( x_0 - \delta \) to \( x_0 + \delta \) lies completely within the practical line.

Then (for \(-\delta < h < \delta\)), the area \( A(x+h) - A(x) \) lies between \((f(x) - \varepsilon)h\) and \((f(x) + \varepsilon)h\), so (for \( h \neq 0 \)),

\[
\frac{A(x + h) - A(x)}{h} \quad \text{lies between} \quad f(x) - \varepsilon \quad \text{and} \quad f(x) + \varepsilon
\]

and so, for \(|h| < \delta\),

\[
\left| \frac{A(x + h) - A(x)}{h} \right| < \delta
\]

ie the embodied idea implies the axiomatic definition.
EMBODIED AREA AND FORMAL Riemann Integration

The area function under the blancmange and the derivative of the area (from Tall, 1991b)
The embodied notions of ‘area’ and ‘area-so-far’ as cognitive roots can support Riemann and even Lebesgue integration.
The area function for the discontinuous function $x - \text{int}(x)$ calculated from 0.

The area function magnified.
INTEGRATING HIGHLY DISCONTINUOUS FUNCTIONS

such as \( f(x) = x \) for \( x \) rational, \( f(x) = 1-x \) for \( x \) irrational.

**Idea:** if \((x_n)\) is a sequence of rationals \( x_n = a_n/b_n \) tending to the real number \( x \), then if \( x \) is rational, the sequence \((x_n)\) is ultimately constant and equal to \( x \) otherwise the denominators \( b_n \) grow without limit.

**Definition:** \( x \) is \((\square,N)\)-rational if the sequence of rationals is computed by the continued fraction method and, as soon as \(|x!-a_n/b_n| < \square\) then \( b_n <!N\).

**Code (for Luiz!)**

Define a function \( \text{rational}(x,e,k) \) which returns \( \text{true} \) or \( \text{false} \) depending on whether \( x \) is an \((e,K)\)-rational or not:

```plaintext
DEFINITION rational(x,e,K)
    r=x/:!a1=0!:!b1=1!:!a2=1!:!b2=0
    REPEAT
        n=INT r/:!d=r−n!:!a=n*a2+a1!:!b=n*b2+b1
        IF !d<>0!THEN
            r=1/d!:!a1=a2!:!b1=b2!:!b2=a!:!b2=b
        UNTIL ABS(a/b−x)<e
        IF b<K THEN return TRUE ELSE return FALSE

Try, say \( \square=10^{-8}, N = 10^4 \) for single precision arithmetic
or \( \square=10^{-16}, N = 10^8 \) for double precision.
```
The (pseudo-) rational area (rational step, midpt)

The (pseudo-) irrational area (random step)
Reflections
Embodied calculus hasn’t happened widely yet. Why not?

Programming numerical algorithms: (pre-1980)

Graphics: (early 1980s), eg using graph-plotting programs

Enactive control (1984) allowing interactive exploration (eg Cabri)

Computer algebra systems (early 80s, generally available in late 80s)

Personal portable tools (eg TI-92, PDAs, portables, iBooks with wireless etc)

Multi-media (1990s)

The World Wide Web (1990s)
Constant innovation caused new ideas to be implemented. Mathematicians naturally wanted the latest and “best” tools.

Computer algebra systems give symbolic power while the power of an enactive interface was still to be fully understood. An embodied approach combines a good human interface into symbolic power. Those focusing mainly on the symbolic rather than the embodied have a great deal of mental reconstruction to do to begin to even understand the power of building on embodied enaction.
Epilogue

- The human mind does not always do mathematics logically, but is guided by a concept image that can be both helpful and also deceptive.
- Symbolism is more precise and safer than visualisation, but cognitive development of symbols in arithmetic, algebra and calculus have many potential cognitive pitfalls.
- *Local straightness* provides an embodied foundation for the calculus.
- The *local slope of the graph* as rate of change is an embodied foundation for the slope function (derivative).
- *Finding a graph given its slope* is an embodied foundation for differential equations.
- *Local flatness* is a cognitive foundation for continuity and the fundamental theorem of calculus.
- An embodied approach can be used to link directly to proceptual symbolism in calculus and to axiomatic formulation in analysis.
What can we do on the TI-92?

1. What not to do.

Experience shows that students using only a symbolic interface learn how to push buttons!


Available as download paper 1994c from
http://www.warwick.ac.uk/staff/David.Tall/downloads.html
or via: http://www.davidtall.com/papers

Review of calculus research as current writing in draft (by Tall, Smith, Piez) also available from same site.

Typical ‘how-to’ procedural, but (sadly) non-conceptual use of the TI-92 can be found on the TI web page:

http://education.ti.com/

or direct to
http://education.ti.com/product/explorations/subs/acloserlook/demo_samples/calculus/democalc_01.html

eg: http://education.ti.com/activity/courses/calculus/jahrc/jahrc03.html

2. What to do: use a graphic approach with flexible software to gain embodied insight, then use the TI-92 to fill out the experience.

I would always support the initial stages with a PC (aat least one for whole class discussion). The embodied ideas need to be there before the sequential exploration by the student on the TI-92.

The original GRAPHIC CALCULUS for PC is a free download from

http://www.graphicccalculus.co.uk
Using the TI-92:

To Zoom in on a graph with equal scales:

Select **WINDOW** and specify a suitable x-range, y-range, say from **-10** to **10** for both, then **F2:ZoomSqr** to reset the x-range to make the graph square.

In **WINDOW** choose **F2:SetFactor**... to set the zoom factors to, say **10**, to zoom in fast.

Use **Y=** to define any function, say **y1(x)=sin(x)**. (Use **F4** to switch off other graphs.) Use **GRAPH** to draw the graph, zoom in with **F2:ZoomIn**.
To draw the slope function
Add the graph:

\[ y_2(x) = \frac{\sin(x+0.001) - \sin(x)}{0.001} \]

to draw the slope function.

Alternatively use 0.001 \textbf{STO} \ h to set \ h to 0.001. and draw the graph:

\[ y_2(x) = \frac{\sin(x+h) - \sin(x)}{h} \]

Better: use \textbf{APPS} 7, Type: function, variable: \ f and type:

\begin{verbatim}
f(x)  :Func  :sin(x)  :Endfunc
\end{verbatim}

Then

\textbf{APPS} 7, Type: function, variable: \slope and type:

\begin{verbatim}
slope(x)  :Func  :(f(x+h)-f(x))/h  :Endfunc
\end{verbatim}

Now draw the two graphs

\[ y_1 = f(x) \]

\[ y_2 = \text{slope}(x) \]

By editing the function \ f(x), and giving \ h various values, you can investigate \textit{any} slope function!
The blancmange function defined on the TI-92

The saw function:

:saw(x)
:Func
:min(x-int(x),1-x+int(x))
:EndFunc

The nth approximation to the blancmange:

:ap(x,n)
:Func
:Local k, s, f
:0 $\leq$ s $: 1$ $\leq$ f
:For k,1,n : s+saw(x)*f $\leq$ s
:2*x - int(x) $\leq$ x : f/2 $\leq$ s
:EndFor
:s
:EndFunc

For practical purposes, use a suitable approximation, eg

:bl(x)
:Func
:ap(x,10)
:EndFunc

Speed of calculation is a TI-92 problem. However, draw

$y_3(x)$ = saw(x)

or

$y_4(x)$ = bl(x)
Continuity on the TI-92

To stretch a graph horizontally:
Select **WINDOW** and specify a suitable x-range, y-range, say from **-10** to **10** for both, then **F2:ZoomSqr** to reset the x-range to make the graph square.
In **WINDOW** choose **F2:SetFactor...** to set

\[ \text{xFact:10, yFact:1} \]

Use **GRAPH** to draw the graph of \( y=\text{abs}(\sin(x)) \), to stretch the graph horizontally using **F2:ZoomIn**.
Try

\[ y=x^{(2/3)}+1 \]

Zooming into (0,1) with **xFact:10, yFact:10** gives a (poor approximation to) a vertical cusp.
Zooming into (0,1) with **xFact:10, yFact:1** gives a horizontal line.